J Glob Optim (2009) 44:159–174
DOI 10.1007/s10898-008-9310-1

Levitin–Polyak well-posedness of variational inequality problems with functional constraints

X. X. Huang · X. Q. Yang · D. L. Zhu

Received: 5 October 2007 / Accepted: 21 April 2008 / Published online: 16 May 2008

Abstract In this paper, we introduce several types of (generalized) Levitin–Polyak well-posednesses for a variational inequality problem with abstract and functional constraints. Criteria and characterizations for these types of well-posednesses are given. Relations among these types of well-posednesses are also investigated.

Keywords Constrained variational inequality · Approximating solution sequence · (generalized) Levitin–Polyak well-posedness · Monotone operator · Coercivity

1 Introduction and preliminaries

Well-posedness of unconstrained and constrained scalar optimization problems was first introduced and studied in Tykhonov [28] and Levitin and Polyak [18], respectively. Since then, various notions of well-posednesses have been defined and extensively studied (see, e.g., [6,8,23,27,32–34]). Recent studies on well-posedness of optimization problems have been extended to vector optimization problems (see, e.g., [4,7,13,14,21,22]). The study of Levitin-Polyak well-posedness for convex scalar optimization problems with functional constraints originates from [17]. Most recently, this research was extended to nonconvex optimization problems with both abstract and functional constraints [15] and nonconvex vector optimization problems with abstract and functional constraints [16]. Tykhonov and Hadamard well-posednesses of (scalar) variational inequality problems, equilibrium problems and mathematical programs with variational inequality (Nash equilibrium) constraints have been studied in the literature (see, e.g., [19,20,24,26,30] and the references therein). Presently, there are two approaches for defining approximating sequences in the study of well-posedness of variational inequality problems. One is based on the gap function given

X. X. Huang (✉) · D. L. Zhu
School of Management, Fudan University, Shanghai 200433, China
e-mail: xxhuang@fudan.edu.cn

X. Q. Yang
Department of Applied Mathematics, Hong Kong Polytechnic University, Kowloon, Hong Kong

by Auslender [1] (see, e.g., [24,26,30]) and the other is based on the gap function given in Fukushima [9] (see, e.g., [19,20]). It is worth noting that all the variational inequaliies considered in [19,20,24,26,30], etc., are defined on abstract sets. It is obvious that variational inequalities with both abstract and explicit constraints are more difficult to handle. Recently, some authors studied augmented Lagrangian methods and penalty methods for variational inequalities with explicit constraints (see, e.g., [2,3,12,31]).

It is easily seen that one can equivalently transform a variational inequality problem with explicit constraints into a minimizing problem with explicit constraints by using the Auslender gap function. As noted in [15], the study of Levitin–Polyak (LP in short) well-posednesses and generalized LP well-posednesses is important in both theory and methodology for a minimizing problem with explicit constraints. In this paper, we will adapt the various versions of LP well-posednesses and generalized LP well-posednesses for constrained minimizing problems to define various types of LP well-posednesses and generalized LP well-posednesses for a variational inequality problem with explicit constraints. We will derive various criteria and characterizations for the various (generalized) LP well-posedness of constrained variational inequalities. Relationships among these well-posednesses will also be established.

Let $(X, \|\cdot\|)$ be a normed space, X^* be its dual space, and (Y, d_1) be a metric space. Let $X_1 \subset X$ and $K \subset Y$ be nonempty and closed sets. Let $F : X_1 \to X^*$ and $g : X_1 \to Y$ be two functions. We denote by $\langle F(x), z\rangle$ the value of the functional $F(x)$ at z.

Let

$$X_0 = \{x \in X_1 : g(x) \in K\}.$$

Consider the following constrained variational inequality problem:

$$\text{(VIP)} \qquad \text{Find } \bar{x} \in X_0 \text{ such that}$$
$$\langle F(\bar{x}), x - \bar{x}\rangle \geq 0, \quad \forall x \in X_0.$$

Throughout the paper, we always assume that $X_0 \neq \emptyset$ and g is continuous on X_1.

Denote by $\bar{X}$ the solution set of (VIP).

Let (P, d) be a metric space and $P_1 \subset P$. We denote by $d_{P_1}(p) = \inf\{d(p, p') : p' \in P_1\}$ the distance from the point $p \in P$ to the set P_1.

Definition 1.1 (i) A sequence $\{x_n\} \subset X_1$ is called a type I LP approximating solution sequence if there exists $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ such that

$$d_{X_0}(x_n) \leq \epsilon_n \tag{1}$$

and

$$\langle F(x_n), x - x_n\rangle \geq -\epsilon_n, \quad \forall x \in X_0. \tag{2}$$

(ii) $\{x_n\} \subset X_1$ is called a type II LP approximating solution sequence if there exist $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ and $\{y_n\} \subset X_0$ satisfying (1), (2) and

$$\langle F(x_n), y_n - x_n\rangle \leq \epsilon_n. \tag{3}$$

(iii) $\{x_n\} \subset X_1$ is called a generalized type I LP approximating solution sequence if there exists $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ satisfying

$$d_K(g(x_n)) \leq \epsilon_n \tag{4}$$

and (2).

(iv) $\{x_n\} \subset X_1$ is called a generalized type II LP approximating solution sequence if there exist $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ and $\{y_n\} \subset X_0$ satisfying (4), (2) and (3).

Definition 1.2 (VIP) is said to be type I (resp. type II, generalized type I, generalized type II) LP well-posed if the solution set $\bar{X}$ of (VIP) is nonempty, and for any type I (resp. type II, generalized type I, generalized type II) LP approximating solution sequence $\{x_n\}$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in \bar{X}$ such that $x_{n_j} \to \bar{x}$.

Remark 1.1 (i) It is clear that any (generalized) type II LP approximating solution sequence is a (generalized) type I LP approximating solution sequence. Thus, (generalized) type I LP well-posedness implies (generalized) type II LP well-posedness.

(ii) Each type of LP well-posedness of (VIP) implies that the solution set $\bar{X}$ is compact.

(iii) Suppose that g is uniformly continuous on a set

$$X_1(\delta_0) = \{x \in X_1 : d_{X_0}(x) \le \delta_0\} \tag{5}$$

for some $\delta_0 > 0$. Then, generalized type I (type II) LP well-posedness of (VIP) implies its type I (type II) LP well-posedness.

To see that the various LP well-posednesses of (VIP) are adaptations of the corresponding LP well-posednesses in minimization problems by using the Auslender gap function, we consider the following general constrained optimization problem:

$$\begin{aligned} \text{(P)} \quad & \min \ f(x) \\ & \text{s.t.} \ x \in X_1, \\ & \quad\ g(x) \in K, \end{aligned}$$

where $f : X_1 \to R^1 \cup \{+\infty\}$ is lower semicontinuos (l.s.c. in short). The feasible set of (P) is still denoted by X_0. The optimal set and optimal value of (P) are denoted by $\bar{X}'$ and $\bar{v}$, respectively. Note that if $\mathrm{Dom}(f) \cap X_0 \neq \emptyset$, where

$$\mathrm{Dom}(f) = \{x \in X_1 : f(x) < +\infty\},$$

then $\bar{v} < +\infty$. In this paper, we always assume that $\bar{v} > -\infty$. LP well-posednesses for the special case where f is finite-valued have been studied in [15].

Definition 1.3 (i) A sequence $\{x_n\} \subset X_1$ is called a type I LP minimizing sequence for (P) if

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v} \tag{6}$$

and

$$d_{X_0}(x_n) \to 0. \tag{7}$$

(ii) $\{x_n\} \subset X_1$ is called a type II LP minimizing sequence for (P) if

$$\lim_{n\to+\infty} f(x_n) = \bar{v} \tag{8}$$

and (7) holds.

(iii) $\{x_n\} \subset X_1$ is called a generalized type I LP minimizing sequence for (P) if (6) holds and

$$d_K(g(x_n)) \to 0. \tag{9}$$

(iv) $\{x_n\} \subset X_1$ is called a generalized type II LP minimizing sequence for (P) if (8) and (9) hold.

Definition 1.4 (P) is said to be type I (resp. type II, generalized type I, generalized type II) LP well-posed if, $\bar{v}$ is finite, $\bar{X}' \neq \emptyset$ and for any type I (resp. type II, generalized type I, generalized type II) LP minimizing sequence $\{x_n\}$ of (P), there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in \bar{X}'$ such that $x_{n_j} \to \bar{x}$.

The Auslender gap function for (VIP) is

$$f(x) = \sup_{x' \in X_0} \langle F(x), x - x' \rangle, \forall x \in X_1. \tag{10}$$

Clearly, f is a function from X_1 to $(-\infty, +\infty]$. Moreover, if there exist $x_0 \in X_0$ and $a \in R^1$ such that $\langle F(x_0), x - x_0 \rangle \geq a, \forall x \in X_0$, then $\mathrm{Dom}(f) \cap X_0 \neq \emptyset$. The next proposition establishes relationships between the various LP well-posednesses of (VIP) and those of (P) with $f(x)$ defined by (10). It is clear that $\bar{X} \neq \emptyset$ if and only if $\bar{X}' \neq \emptyset$.

Proposition 1.1 *Assume that $\bar{X} \neq \emptyset$. Then, (VIP) is type I (resp. type II, generalized type I, generalized type II) LP well-posed if and only if (P) is type I (resp. type II, generalized type I, generalized type II) LP well-posed with $f(x)$ defined by* (10).

Proof It is well-known that, if $\bar{X} \neq \emptyset$, then $\bar{x}$ is a solution of (VIP) if and only if $\bar{x}$ is an optimal solution of (P) with $\bar{v} = f(\bar{x}) = 0$ and $f(x)$ defined by (10). It is also routine to check that a sequence $\{x_n\}$ is a type I (resp. type II, generalized type I, generalized type II) LP approximating solution sequence of (VIP) if and only if it is a type I (resp. type II, generalized type I, generalized type II) LP minimizing sequence of (P). It follows that (VIP) is type I (resp. type II, generalized type I, generalized type II) LP well-posed if and only if (P) is type I (resp. type II, generalized type I, generalized type II) LP well-posed with $f(x)$ defined by (10). □

To end this section, we note that all the results in [15] for the well-posedness hold for (P) as well so long as $\mathrm{Dom}(f) \cap X_0 \neq \emptyset$.

2 Criteria and characterizations for (generalized) LP well-posedness of (VIP)

In this section, we give necessary and sufficient conditions for the various types of (generalized) LP well-posednesses defined in Sect. 1.

Consider the following statement:

$$[\bar{X} \neq \emptyset \text{ and, for any type I (resp. type II, generalized type I, generalized type II) LP approximating solution sequence } \{x_n\}, \text{ we have } d_{\bar{X}}(x_n) \to 0.] \tag{11}$$

It is elementary to prove the proposition below.

Proposition 2.1 *If (VIP) is type I (resp. type II, generalized type I, generalized type II) LP well-posed, then* (11) *holds. Conversely, if* (11) *holds and $\bar{X}$ is compact, then (VIP) is type I (resp. type II, generalized type I, generalized type II) LP well-posed.*

Now consider a real-valued function $c = c(t, s)$ defined for $t, s \geq 0$ sufficiently small, such that

$$c(t, s) \geq 0, \quad \forall t, s, \quad c(0, 0) = 0, \tag{12}$$

$$s_n \to 0, t_n \geq 0, c(t_n, s_n) \to 0 \text{ imply } t_n \to 0. \tag{13}$$

We state ([15], Theorem 2.1) as the following lemma.

Lemma 2.1 *Consider the constrained optimization problem* (*P*). *If* (*P*) *is type II LP well-posed, then there exists a function c satisfying* (12) *and* (13) *such that*

$$|f(x) - \bar{v}| \geq c(d_{\bar{X}'}(x), d_{X_0}(x)), \quad \forall x \in X_1. \tag{14}$$

Conversely, suppose that $\bar{X}'$ *is nonempty and compact, and (14) holds for some c satisfying (12) and (13). Then (P) is type II LP well-posed.*

The following theorem follows immediately from Proposition 1.1 and Lemma 2.1 with $f(x)$ defined by (10) and $\bar{v} = 0$.

Theorem 2.1 *If* (*VIP*) *is type II LP well-posed, then there exists a function c satisfying* (12) *and* (13) *such that*

$$|f(x)| \geq c(d_{\bar{X}}(x), d_{X_0}(x)), \quad \forall x \in X_1, \tag{15}$$

where $f(x)$ *is defined by* (10). *Conversely, suppose that* $\bar{X}$ *is nonempty and compact, and* (15) *holds for some c satisfying* (12) *and* (13). *Then* (*VIP*) *is type II LP well-posed.*

Analogously, we can establish the following theorem by applying Proposition 1.1 and ([15], Theorem 2.2).

Theorem 2.2 *If* (*VIP*) *is generalized type II LP well-posed, then there exists a function c satisfying* (12) *and* (13) *such that*

$$|f(x)| \geq c(d_{\bar{X}}(x), d_K(g(x))), \quad \forall x \in X_1, \tag{16}$$

where $f(x)$ *is defined by* (10). *Conversely, suppose that* $\bar{X}$ *is nonempty and compact, and* (16) *holds for some c satisfying* (12) *and* (13). *Then* (*VIP*) *is generalized type II LP well-posed.*

Next we give Furi–Vignoli type characterizations [10] for the (generalized) type I LP well-posednesses of (VIP). To this purpose, first we consider the constrained optimization problem (P).

Let $(X, \|\cdot\|)$ be a Banach space. Recall that the Kuratowski measure of noncompactness for a subset A of X is defined as

$$\alpha(A) = \inf\{\epsilon > 0 : A \subset \underset{1 \leq i \leq n}{\cup} C_i, \quad \text{for some } C_i, \text{diam}(C_i) \leq \epsilon\},$$

where $\text{diam}(C_i)$ is the diameter of C_i defined by

$$\text{diam}(C_i) = \sup\{\|x_1 - x_2\| : x_1, x_2 \in C_i\}.$$

Given two nonempty subsets A and B of X, define the excess of set A to set B by

$$e(A, B) = \sup\{d_B(a) : a \in A\}.$$

The Hausdorff distance between A and B is defined as

$$\text{haus}(A, B) = \max\{e(A, B), e(B, A)\}.$$

Let, for each $\epsilon > 0$,

$$\Omega_1(\epsilon) = \{x \in X_1 : f(x) \leq \bar{v} + \epsilon, d_{X_0}(x) \leq \epsilon\},$$
$$\Omega_2(\epsilon) = \{x \in X_1 : f(x) \leq \bar{v} + \epsilon, d_K(g(x)) \leq \epsilon\}.$$

The next lemma can be proved analogously to ([17], Theorem 5.5).

Lemma 2.2 *Let* $(X, \|\cdot\|)$ *be a Banach space. Consider the optimization problem* (*P*). *Suppose that* f *is bounded below on* X_0. *Then,* (*P*) *is* (*generalized*) *type I LP well-posed if and only if*

$$(\lim_{\epsilon\to 0} \alpha(\Omega_2(\epsilon)) = 0) \quad \lim_{\epsilon\to 0} \alpha(\Omega_1(\epsilon) = 0.$$

Now let $f(x)$ *be defined by* (10) *and* $\bar{v} = 0$. *Then, it is easily seen that*

$$\Omega_1(\epsilon) = \{x \in X_1 : \langle F(x), x' - x\rangle \geq -\epsilon, \quad \forall x' \in X_0, d_{X_0}(x) \leq \epsilon\} \tag{17}$$

and

$$\Omega_2(\epsilon) = \{x \in X_1 : \langle F(x), x' - x\rangle \geq -\epsilon, \quad \forall x' \in X_0, d_K(g(x)) \leq \epsilon\}. \tag{18}$$

Theorem 2.3 *Let* $(X, \|\cdot\|)$ *be a Banach space. Assume that* $\bar{X} \neq \emptyset$. *Let* $\Omega_1(\epsilon)$ *and* $\Omega_2(\epsilon)$ *be defined by* (17) *and* (18), *respectively. Then,* (*VIP*) *is* (*generalized*) *type I LP well-posed if and only if*

$$(\lim_{\epsilon\to 0} \alpha(\Omega_2(\epsilon)) = 0) \quad \lim_{\epsilon\to 0} \alpha(\Omega_1(\epsilon) = 0.$$

Proof Note that the function $f(x)$ defined by (10) is nonnegative on X_0. By Proposition 1.1 and Lemma 2.2, the conclusion follows. □

Definition 2.1 (i) Let Z be a topological space and $Z_1 \subset Z$ be nonempty. Suppose that $h : Z \to R^1 \cup \{+\infty\}$ is an extended real-valued function. h is said to be level-compact on Z_1 if, for any $s \in R^1$, the subset $\{z \in Z_1 : h(z) \leq s\}$ is compact.

(ii) Let Z be a finite dimensional normed space and $Z_1 \subset Z$ be nonempty. A function $h : Z \to R^1 \cup \{+\infty\}$ is said to be level-bounded on Z_1 if Z_1 is bounded or

$$\lim_{z\in Z_1, \|z\|\to+\infty} h(z) = +\infty.$$

The following proposition presents some sufficient conditions for type I LP well-posedness of (VIP).

Proposition 2.2 *Assume that for each* $x' \in X_0$, $\langle F(\cdot), x' - (\cdot)\rangle$ *is upper semicontinuous* (*u.s.c. in short*) *on the set* $X_1(\delta_0)$, *which is defined by* (5). *Suppose that the solution set* $\bar{X}$ *of* (*VIP*) *is nonempty. Further assume that one of the following conditions holds.*

(*i*) *There exists* $0 < \delta_1 \leq \delta_0$ *such that* $X_1(\delta_1)$ *is compact, where*

$$X_1(\delta_1) = \{x \in X_1 : d_{X_0}(x) \leq \delta_1\}; \tag{19}$$

(*ii*) *the function* f *defined by* (10) *is level-compact on* X_1;

(*iii*) X *is finite dimensional and*

$$\lim_{x\in X_1, \|x\|\to+\infty} max\{f(x), d_{X_0}(x)\} = +\infty, \tag{20}$$

where f *is defined by* (10);

(*iv*) *there exists* $0 < \delta_1 \leq \delta_0$ *such that* f *is level-compact on* $X_1(\delta_1)$ *defined by* (19).

Then, (VIP) is type I LP well-posed.

Proof Let $\{x_n\}$ be a type I LP approximating solution sequence. Then, there exists $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ such that

$$\langle F(x_n), x' - x_n \rangle \geq -\epsilon_n, \forall x' \in X_0, \tag{21}$$

$$d_{X_0}(x_n) \leq \epsilon_n. \tag{22}$$

(i) From (22), we can assume without loss of generality that $\{x_n\} \subset X_1(\delta_1)$. By the compactness of $X_1(\delta_1)$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in X_1(\delta_1)$ such that $x_{n_j} \to \bar{x}$. From this fact and (22), we have $\bar{x} \in X_0$. Furthermore, by (21), we have

$$\langle F(x_{n_j}), x' - x_{n_j} \rangle \geq -\epsilon_{n_j}, \quad \forall x' \in X_0.$$

Passing to the upper limit, we get

$$\langle F(\bar{x}), x' - \bar{x} \rangle \geq \lim sup_{j\to+\infty} \langle F(x_{n_j}), x' - x_{n_j} \rangle \geq 0, \quad \forall x' \in X_0.$$

Thus, $\bar{x} \in \bar{X}$.

It is obvious that (ii) implies (iv). Now we show that (iii) implies (iv). Indeed, we need only to show that for any $t \in R^1$, the set

$$A = \{x \in X_1(\delta_1) : f(x) \leq t\}$$

is bounded since X is a finite dimensional space and the function f defined by (10) is l.s.c. on $X_1(\delta_0)$ (by the u.s.c. of $\langle F(\cdot), x' - (\cdot) \rangle, \forall x' \in X_0$) and thus, A is closed. Suppose to the contrary that there exist $t \in R^1$ and $\{x'_n\} \subset X_1(\delta_1)$ such that $\|x'_n\| \to +\infty$ and $f(x'_n) \leq t$. From $\{x'_n\} \subset X_1(\delta_1)$, we have

$$d_{X_0}(x'_n) \leq \delta_1.$$

Thus,

$$\max\{f(x'_n), d_{X_0}(x'_n)\} \leq \max\{t, \delta_1\},$$

contradicting (20). Consequently, we need only to prove that if (iv) holds, then (VIP) is type I LP well-posed.

(iv) By (22), we can obviously assume without loss of generality that $\{x_n\} \subset X_1(\delta_1)$. From (21), we can assume without loss of generality that

$$\{x_n\} \subset \{x \in X_1 : f(x) \leq 1\},$$

where f is defined by (10). By the level-compactness of f on $X_1(\delta_1)$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in X_1(\delta_1)$ such that $x_{n_j} \to \bar{x}$. From this fact and (22), we have $\bar{x} \in X_0$. Furthermore, by (21), we have

$$\langle F(x_{n_j}), x' - x_{n_j} \rangle \geq -\epsilon_{n_j}, \quad \forall x' \in X_0.$$

Passing to the upper limit, we get

$$\langle F(\bar{x}), x' - \bar{x} \rangle \geq \lim sup_{j\to+\infty} \langle F(x_{n_j}), x' - x_{n_j} \rangle \geq 0, \forall x' \in X_0.$$

Thus, $\bar{x} \in \bar{X}$. □

Similarly, we can prove the next proposition.

Proposition 2.3 *Assume that there exists $\delta_0 > 0$ such that for each $x' \in X_0$, $\langle F(\cdot), x' - (\cdot)\rangle$ is u.s.c. on the set*

$$X_2(\delta_0) = \{x \in X_1 : d_K(g(x)) \le \delta_0\}.$$

Suppose that the solution set $\bar{X}$ of (VIP) is nonempty. Further assume that one of the following conditions holds.

(*i*) *There exists $0 < \delta_1 \le \delta_0$ such that $X_2(\delta_1)$ is compact, where*

$$X_2(\delta_1) = \{x \in X_1 : d_K(g(x)) \le \delta_1\}; \tag{23}$$

(*ii*) *the function f defined by (10) is level-compact on X_1;*
(*iii*) *X is finite dimensional and*

$$\lim_{x \in X_1, \|x\| \to +\infty} max\{f(x), d_K(g(x))\} = +\infty,$$

where f is defined by (10);
(*iv*) *there exists $0 < \delta_1 \le \delta_0$ such that f is level-compact on $X_2(\delta_1)$ defined by (23).*

Then, (VIP) is generalized type I well-posed.

Remark 2.1 If F is continuous on X_1, then the function $\langle F(\cdot), x' - (\cdot)\rangle$ is continuous on X_1 for any $x' \in X_0$.

Proposition 2.4 *Let X be finite dimensional. Let F be continuous on X_1 and the solution set $\bar{X}$ of (VIP) be nonempty. Suppose that there exist $\delta_1 > 0$ and $x_0 \in X_0$ such that the function $\langle F(x), x - x_0\rangle$ is level-bounded on the set $X_1(\delta_1)$ defined by (19). Then, (VIP) is type I LP well-posed.*

Proof Let $\{x_n\}$ be a type I LP approximating solution sequence. Then, there exists $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ such that

$$\langle F(x_n), x' - x_n\rangle \ge -\epsilon_n, \quad \forall x' \in X_0, \tag{24}$$
$$d_{X_0}(x_n) \le \epsilon_n. \tag{25}$$

From (25), we can assume without loss of generality that $\{x_n\} \subset X_1(\delta_1)$. Let us show by contradiction that $\{x_n\}$ is bounded. Otherwise, we assume without loss of generality that $\|x_n\| \to +\infty$. By the level-boundedness condition, we have

$$\lim_{n \to +\infty} \langle F(x_n), x_0 - x_n\rangle = -\infty,$$

contradicting (24) (with x' replaced by x_0) when n is sufficiently large. Consequently, we can assume without loss of generality that $x_n \to \bar{x}$. Obviously, $\bar{x} \in X_0$. Furthermore, taking the limit in (24), we see that $\bar{x} \in \bar{X}$. □

Similarly, we can prove the next result.

Proposition 2.5 *Let X be finite dimensional. Let F be continuous on X_1 and the solution set $\bar{X}$ of (VIP) be nonempty. Suppose that there exist $\delta_1 > 0$ and $x_0 \in X_0$, the function $\langle F(x), x - x_0\rangle$ is level-bounded on the set $X_2(\delta_1)$ defined by (23). Then, (VIP) is generalized type I LP well-posed.*

Let X be finite dimensional and $X_3 \subset X$ be nonempty. Recall that $F : X \to X^*$ is said to be strongly coercive on X_3 (cf. [29]) *if X_3 is bounded or*

$$\lim_{x \in X_3, \|x\| \to +\infty} \frac{\langle F(x), x - x' \rangle}{\|x - x'\|} = +\infty$$

holds for any $x' \in X_3$. Clearly, F is strongly coercive on $X_1(\delta_1)$ (defined by (19)) implies that for each $x' \in X_0$, the function $\langle F(x), x - x' \rangle$ is level-bounded on $X_1(\delta_1)$; F is strongly coercive on $X_2(\delta_1)$ (defined by (23)) implies that for each $x' \in X_0$, the function $\langle F(x), x - x' \rangle$ is level-bounded on $X_2(\delta_2)$. Thus, we have the next two corollaries.

Corollary 2.1 *Let X be finite dimensional. Let F be continuous on X_1 and the solution set $\bar{X}$ of (VIP) be nonempty. Suppose that there exists $\delta_1 > 0$ such that F is strongly coercive on $X_1(\delta_1)$ defined by (19). Then, (VIP) is type I LP well-posed.*

Corollary 2.2 *Let X be finite dimensional. Let F be continuous on X_1 and the solution set $\bar{X}$ of (VIP) be nonempty. Suppose that there exists $\delta_1 > 0$ such that F is strongly coercive on $X_2(\delta_1)$ defined by (23). Then, (VIP) is generalized type I LP well-posed.*

Definition 2.2 [11] Let X be finite dimensional and $X_3 \subset X$.

(i) F is said to be monotone on X_3 if

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in X_3.$$

(ii) F is said to be coercive on X_3 if X_3 is bounded or there exists $x_0 \in X_3$ such that

$$\lim_{x \in X_3, \|x\| \to +\infty} \frac{\langle F(x), x - x_0 \rangle}{\|x\|} = +\infty.$$

The following proposition establishes type I LP well-posedness under monotonicity and coercivity of the mapping F.

Proposition 2.6 *Let X be finite dimensional. Let F be continuous on X_1 and the solution set $\bar{X}$ of* (*VIP*) *be nonempty. Suppose that there exists $\delta_1 > 0$ such that F is monotone and coercive on $X_1(\delta_1)$ defined by* (19)*. Further assume that the set*

$$Q = \{x \in X_0 : \quad \text{for any } x_1 \in X_1(\delta_1), \text{ there exists } t_0 \in (0, 1) \text{ such that } x_1 + t_0(x - x_1) \in X_0\} \tag{26}$$

is nonempty. Then, (*VIP*) *is type I LP well-posed.*

Proof Let $\{x_n\}$ be a type I LP approximating solution sequence. Then, there exists $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ such that

$$\langle F(x_n), x' - x_n \rangle \geq -\epsilon_n, \quad \forall x' \in X_0, \tag{27}$$

$$d_{X_0}(x_n) \leq \epsilon_n. \tag{28}$$

From (28), we can assume without loss of generality that $\{x_n\} \subset X_1(\delta_1)$. Let us show by contradiction that $\{x_n\}$ is bounded. Suppose that $x'_0 \in Q$. Then, there exists $t_0 \in (0, 1)$ such that $x_0 + t_0(x'_0 - x_0) \in X_0$, where $x_0 \in X_1(\delta_1)$ is the element in the definition of coercivity of F on $X_1(\delta_1)$. Thus, from (27), we have

$$\langle F(x_n), x_0 + t_0(x'_0 - x_0) - x_n \rangle \geq -\epsilon_n.$$

That is,

$$(1-t_0)\langle F(x_n), x_0 - x_n\rangle + t_0\langle F(x_n), x_0' - x_n\rangle \geq -\epsilon_n.$$

Namely,

$$(1-t_0)\langle F(x_n), x_0 - x_n\rangle + t_0\left[\langle F(x_n) - F(x_0'), x_0' - x_n\rangle + \langle F(x_0'), x_0' - x_n\rangle\right] \geq -\epsilon_n. \tag{29}$$

By the monotonicity of F on $X_1(\delta_1)$, we have

$$\langle F(x_n) - F(x_0'), x_0' - x_n\rangle \leq 0. \tag{30}$$

The combination of (29) and (30) yields

$$(1-t_0)\langle F(x_n), x_0 - x_n\rangle + t_0\langle F(x_0', x_0' - x_n\rangle \geq -\epsilon_n. \tag{31}$$

If $\{x_n\}$ is unbounded, we assume without loss of generality that $\|x_n\| \to +\infty$. Then, from the coercivity of F on $X_1(\delta_1)$, we have

$$\lim_{n\to+\infty} \frac{\langle F(x_n), x_0 - x_n\rangle}{\|x_n\|} = -\infty.$$

Furthermore, it is obvious that $\left\{\frac{\langle F(x_0'), x_0' - x_n\rangle}{\|x_n\|}\right\}$ is bounded. Dividing (31) by $\|x_n\|$ and passing to the limit, a contradiction arises. Hence, $\{x_n\}$ is bounded. Thus, we can find a subsequence $\{x_{n_j}\}$ and $\bar{x} \in X_1(\delta_1)$ such that $x_{n_j} \to \bar{x}$. Taking the limit in (28) (with x_n replaced by x_{n_j}), we have $\bar{x} \in X_0$. Taking the limit in (27) (with x_n replaced by x_{n_j}), we obtain

$$\langle F(\bar{x}), x - \bar{x}\rangle \geq 0, \forall x \in X_0.$$

That is, $\bar{x} \in \bar{X}$. The proof is complete. □

Similarly, we can prove the next result.

Proposition 2.7 *Let X be finite dimensional. Let F be continuous on X_1 and the solution set $\bar{X}$ of (VIP) be nonempty. Suppose that there exists $\delta_1 > 0$ such that F is monotone and coercive on $X_2(\delta_1)$ defined by (23). Further assume that the set Q defined by (26) (with $X_1(\delta_1)$ replaced by $(X_2(\delta_1))$ is nonempty. Then, (VIP) is generalized type I LP well-posed.*

Remark 2.2 If $\text{int}X_0 \neq \emptyset$, then it is obvious that the set Q defined by (26) is nonempty.

Now we consider the case when Y is a normed space, K is a closed and convex cone with nonempty interior $\text{int}K$ and let $e \in \text{int}K$.

Let $t \geq 0$ and denote

$$X_4(t) = \{x \in X_1 : g(x) \in K - te\}. \tag{32}$$

Proposition 2.8 *Let Y be a normed space, K be a closed and convex cone with nonempty interior $\text{int}K$ and $e \in \text{int}K$. Assume that there exists $t_1 > 0$ such that*

(i) for each $x' \in X_0$, $\langle F(\cdot), x' - (\cdot)\rangle$ is u.s.c. on $X_4(t_1)$;
(ii) the function $f(x)$ defined by (10) is level-compact on $X_4(t_1)$.

Further assume that the solution set $\bar{X}$ of (VIP) is nonempty. Then, (VIP) is generalized type I LP well-posed.

Proof Let $\{x_n\}$ be a generalized type I LP approximating solution sequence. Then, there exists $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ such that

$$\langle F(x_n), x' - x_n \rangle \geq -\epsilon_n, \forall x' \in X_0, \tag{33}$$

$$d_K(g(x_n)) \leq \epsilon_n. \tag{34}$$

From (34), we deduce that there exists $\{k_n\} \subset K$ such that

$$\|g(x_n) - k_n\| \leq 2\epsilon_n.$$

That is,

$$g(x_n) - k_n \in 2\epsilon_n B, \tag{35}$$

where B is the closed unit ball of Y. We assert that there exists $M_0 > 0$ such that

$$B \subset K - M_0 e. \tag{36}$$

Otherwise, there exist $b_n \in B$ and $0 < M_n \to +\infty$ such that

$$b_n + M_n e \notin K, \quad \forall n.$$

Thus,

$$b_n + M_n e \notin \text{int}K, \quad \forall n,$$

implying

$$b_n/M_n + e \notin \text{int}K, \quad \forall n. \tag{37}$$

Taking the limit in (37), we get $e \notin \text{int}K$, contradicting the assumption. It follows from (35) and (36) that

$$g(x_n) - k_n \in K - 2M_0\epsilon_n e.$$

Thus,

$$g(x_n) \in K - 2M_0\epsilon_n e.$$

Consequently, we can assume without loss of generality that

$$x_n \in X_4(t_1) \tag{38}$$

since $2M_0\epsilon_n \to 0$ as $n \to +\infty$. From (33), we have

$$f(x_n) \leq \epsilon_n, \forall n. \tag{39}$$

From (38), (39) and the level-compactness of f on $X_4(t_1)$, we see that there exist a subsequence $\{x_{n_j}\}$ and $\bar{x} \in X_4(t_1)$ such that $x_{n_j} \to \bar{x}$. Taking the limit in (34) (with n replaced by n_j), we get $\bar{x} \in X_0$. Taking the upper limit in (33) (with n replaced by n_j), we obtain

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in X_0.$$

Thus, $\bar{x} \in \bar{X}$. The proof is complete. □

Definition 2.3 $X_1 \subset X$ is said to have property (P_1) if for any $x_1, x_2 \in X_1$, there exists $t_0 \in (0, 1)$ such that $x_1 + t(x_2 - x_1) \in X_1, \forall t \in (0, t_0)$.

Proposition 2.9 *Let X be finite dimensional. Let Y be a normed space, K be a closed and convex cone with nonempty interior $intK$ and $e \in intK$. Let F be continuous on X_1. Suppose that there exists $t_1 > 0$ such that F is monotone and coercive on $X_4(t_1)$ given by (32). Further assume that X_1 has property (P_1), there exists $x_0' \in X_1$ such that $g(x_0') \in intK$ and the solution set $\bar{X}$ of (VIP) is nonempty. Then, (VIP) is generalized type I LP well-posed.*

Proof Let $\{x_n\}$ be a generalized type I LP approximating solution sequence. Then, there exists $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n \to 0$ such that

$$\langle F(x_n), x' - x_n\rangle \geq -\epsilon_n, \quad \forall x' \in X_0,$$

$$d_K(g(x_n)) \leq \epsilon_n. \tag{40}$$

Arguing as in the proof of Proposition 2.8, we can deduce from (40) that $x_n \in X_4(t_1)$ when n is sufficiently large. As $g(x_0') \in \text{int}K$, g is continuous on X_1 and X_1 has property (P_1), we can easily see that the set Q defined by (26) (with $X_1(\delta_1)$ replaced by $X_4(t_1)$) is nonempty. In fact, $x_0' \in Q$. The rest of the proof is the same as that of Proposition 2.6 (with $X_1(\delta_1)$ replaced by $X_4(t_1)$). □

Now we consider the case when X is a finite dimensional normed space, and X_0 and $X_1 \subset X$ are convex. Let

$$h(x) = \sup_{x' \in X_0} \langle F(x'), x - x'\rangle, \quad \forall x \in X_1.$$

Clearly, $h(x)$ is a convex function on X_1. Moreover, if F is pseudomonotone on X_0, i.e.,

$$\langle F(x'), x - x'\rangle \geq 0 \Rightarrow \langle F(x), x - x'\rangle \geq 0, \quad \forall x, x' \in X_0,$$

then $h(x)$ is nonnegative on X_0 and vanishes at any solution of (VIP). On the other hand, any minimizer $\bar{x}$ of h over X_0 with $h(\bar{x}) = 0$ is a solution of (VIP) if F is continuous on X_0. This implies that (VIP) is equivalent to minimizing $h(x)$ over X_0 ([25], Proposition 2.1) if F is pseudomonotone and continuous on X_0 and $\bar{X} \neq \emptyset$. That is, the optimal set of minimizing $h(x)$ over X_0 is also $\bar{X}$.

It is obvious that the above conclusion also holds if F is monotone and continuous on X_0 and $\bar{X} \neq \emptyset$.

We make the following assumption.

Assumption 2.1 X is a finite dimensional normed space, Y is a normed space, $X_1 \subset X$ is a nonempty, closed and convex set. $K \subset Y$ is a closed and convex cone with nonempty interior $intK$, F is continuous and monotone on X_1, g is concave on X_1 (i.e., for any $x_1, x_2 \in X_1$ and any $\theta \in (0, 1)$, there holds that $g(\theta x_1 + (1-\theta)x_2) - \theta g(x_1) - (1-\theta)g(x_2) \in K$). The solution set of (VIP) is nonempty.

It is obvious that under Assumption 2.1, the optimization problem (P) (with f replaced by h) is a convex program.

We need the following lemma, whose proof is elementary and thus omitted.

Lemma 2.3 *Let F be monotone on X_1 and $\bar{X} \neq \emptyset$. Then, if $\{x_n\}$ is a (generalized) type I LP approximating solution sequence of (VIP), then it is a (generalized) type I LP minimizing sequence of (P) (with f replaced by h).*

We state Theorem 2.4 of [15] as the following lemma.

Lemma 2.4 *Let Assumption 2.1 hold. Then,* (P) (*with* f *replaced by* h) *is generalized type I LP well-posed if and only if the optimal set* $\bar{X}$ *is nonempty and compact.*

Similar to the proof of ([15], Theorem 2.4), we can prove the following lemma.

Lemma 2.5 *Let Assumption 2.1 hold. Then,* (P) (*with* f *replaced by* h) *is type I LP well-posed if and only if the optimal set of* (P) *is nonempty and compact.*

The following theorem is a direct consequence of Lemmas 2.3–2.5 and (i) and (ii) of Remark 1.1.

Theorem 2.4 *Let Assumption 2.1 hold. Then, any type of* (*generalized*) *LP well-posedness of* (*VIP*) *is equivalent to the fact that the solution set* $\bar{X}$ *of* (*VIP*) *is nonempty and compact.*

3 Relations among Various (Generalized) LP Well-Posednesses

Simple relationships among the (generalized) LP well-posednesses were mentioned in Remark 1.1. Under Assumption 2.1, the equivalence of all types of (generalized) LP well-posednesses was established in Theorem 2.4. In this section, we investigate further relationships among them. First, we state ([15], Theorem 3.1) as the following lemma.

Lemma 3.1 *Consider the optimization problem (P). Suppose that there exist* $\delta > 0$, $\alpha > 0$ *and* $c > 0$ *such that*

$$d_{X_0}(x) \leq c d_K^{\alpha}(g(x)), \quad \forall x \in X_2(\delta), \tag{41}$$

where

$$X_2(\delta) = \{x \in X_1 : d_K(g(x)) \leq \delta\}. \tag{42}$$

If (P) is type I (type II) LP well-posed, then (P) is type I (type II) LP well-posed in the generalized sense.

The following result follows immediately from Proposition 1.1 and Lemma 3.1.

Theorem 3.1 *If there exist* $\delta > 0$, $\alpha > 0$ *and* $c > 0$ *such that* (41) *holds, then the type I* (*type II*) *LP well-posedness of* (*VIP*) *implies its generalized type I* (*generalized type II*) *LP well-posedness.*

Definition 3.1 [5] Let W be a topological space and $F : W \to 2^X$ be a set-valued map. F is said to be upper Hausdorff semicontinuous (u.H.c. in short) at $w \in W$ if, for any $\epsilon > 0$, there exists a neighbourhood U of w such that $F(U) \subset B(F(w), \epsilon)$, where, for $Z \subset X$ and $r > 0$,

$$B(Z, r) = \{x \in X : d_Z(x) \leq r\}.$$

Clearly, $X_2(\delta)$ given by (42) can be seen as a set-valued map from R^1_+ to X.
The next lemma is just Theorem 3.2 of [15].

Lemma 3.2 *Assume that the set-valued map* $X_2(\delta)$ *defined by* (42) *is u.H.c. at* $0 \in R^1_+$. *If* (P) *is type I* (*type II*) *LP well-posed, then* (P) *is type I* (*type II*) *LP well-posed in the generalized sense.*

The following theorem is a direct consequence of Proposition 1.1 and Lemma 3.2.

Theorem 3.2 *Suppose that the set-valued map $X_2(\delta)$ defined by* (42) *is u.H.c. at $0 \in R^1_+$. If* (*VIP*) *is type I* (*type II*) *LP well-posed, then* (*VIP*) *is type I* (*type II*) *LP well-posed in the generalized sense.*

Now we consider the case when Y is a normed space.

Lemma 3.3 *Let Y be a normed space and $\{x_n\} \subset X_1$. Then, $d_K(g(\mathrm{x_n})) \to 0$ if and only if there exists $\{y_n\} \subset Z$ with $y_n \to 0$ such that $g(x_n) \in K + y_n, \forall n$.*

Proof Necessity. From $d_K(g(x_n)) \to 0$, we deduce that there exists $\{u_n\} \subset K$ such that

$$\|g(x_n) - u_n\| \to 0.$$

Let $y_n = g(x_n) - u_n$. Then, $y_n \to 0$ and $g(x_n) \in K + y_n$.

Sufficiency. Since $g(x_n) - y_n \in K$,

$$d_K(g(x_n)) \leq \|g(x_n) - (g(x_n) - y_n)\| = \|y_n\| \to 0.$$

□

Let

$$X_5(y) = \{x \in X_1 : g(x) \in K + y\}, \forall y \in Y. \tag{43}$$

Clearly, $X_5(y)$ can be seen as a set-valued map from Y to X.

We state ([15], Theorem 3.5) as the next lemma.

Lemma 3.4 *Assume that the set-valued map $X_5(y)$ defined by* (43) *is u.H.c. at $0 \in Y$. If* (*P*) *is type I* (*type II*) *LP well-posed, then* (*P*) *is type I* (*type II*) *LP well-posed in the generalized sense.*

The next theorem follows immediately from Proposition 1.1, Lemmas 3.3 and 3.4.

Theorem 3.3 *Assume that the set-valued map $X_5(y)$ defined by* (43) *is u.H.c. at $0 \in Y$. If* (*VIP*) *is type I* (*type II*) *LP well-posed, then* (*VIP*) *is type I* (*type II*) *LP well-posed in the generalized sense.*

In the special case when K is a closed and convex cone with nonempty interior $int K$ and $e \in int K$. We consider $X_4(t)$ defined by (32) as a set-valued map from R^1_+ to X. We have the next lemma, which is just Theorem 3.8 of [15].

Lemma 3.5 *Assume that the set-valued map $X_4(t)$ defined by* (32) *is u.H.c. at $0 \in R^1_+$. If* (*P*) *is type I* (*type II*) *LP well-posed, then* (*P*) *is type I* (*type II*) *LP well-posed in the generalized sense.*

The following theorem follows directly from Proposition 1.1 and Lemma 3.5.

Theorem 3.4 *Assume that the set-valued map $X_4(t)$ defined by* (32) *is u.H.c. at $0 \in R^1_+$. If* (*VIP*) *is type I* (*type II*) *LP well-posed, then* (*VIP*) *is type I* (*type II*) *LP well-posed in the generalized sense.*

Acknowledgements This work is supported by the Research Grants Council (BQ-890) of Hong Kong, the National Science Foundation of China and Shanghai Pujiang Program.

References

1. Auslender, A.: Optimization: Methodes Numeriques. Masson, Paris (1976)
2. Auslender, A.: Asymptotic analysis for penalty and barrier methods in variational inequalities. SIAM J. Control Optim. **37**, 653–671 (1999)
3. Auslender, A.: Variational inequalities over the cone of semidefinite positive symmetric matrices and over the Lorentz cone, the Second Japanese-Sino Optimization Meeting, Part II, Kyoto, 2002. Optim. Methods Softw. **18**, 359–376 (2003)
4. Bednarczuk, E.: Well-posedness of Vector Optimization Problems. Lecture Notes in Economics and Mathematical Systems, vol. 294, pp. 51–61. Berlin, Springer-Verlag (1987)
5. Bednarczuk, E., Penot, P.J.: Metrically well-set minimization problems. Appl. Math. Optim. **26**, 273–285 (1992)
6. Beer, G., Lucchetti, R.: The epi-distance topology: continuity and stability results with application to convex optimization problems. Math. Oper. Res. **17**, 715–726 (1992)
7. Deng, S.: Coercivity properties and well-posedness in vector optimization. RAIRO Oper. Res. **37**, 195–208 (2003)
8. Dontchev, A.L., Zolezzi, T.: Well-posed Optimization Problems. Lecture Notes in Mathematics, vol. 1543. Springer, Berlin (1993)
9. Fukushima, M.: Equivalent differentiable optimization problem and descent method for symmteric variational inequalities. Math. Program. **53**, 99–110 (1992)
10. Furi, M., Vignoli, A.: About well-posed minimization problems for functionals in metric spaces. J. Optim. Theory Appl. **5**, 225–229 (1970)
11. Harker, P.T., Pang, J.S.: Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. Math. Program. **48**, 161–220 (1990)
12. He, B.S., Yang, H., Zhang, C.S.: A modified augmented Lagrangian method for a class of monotone variational inequalities. Eur. J. Oper. Res. **159**, 35–51 (2004)
13. Huang, X.X.: Extended well-posed properties of vector optimization problems. J. Optim. Theory Appl. **106**, 165–182 (2000)
14. Huang, X.X.: Extended and strongly extended well-posedness of set-valued optimization problems. Math. Methods Oper. Res. **53**, 101–116 (2001)
15. Huang, X.X., Yang, X.Q.: Generalized Levitin–Polyak well-posedness in constrained optimization. SIAM J. Optim. **17**, 243–258 (2006)
16. Huang, X.X., Yang, X.Q.: Levitin–Polyak well-posedness of constrained vector optimization problems. J. Glob. Optim. Published Online, 06 July (2006)
17. Konsulova, A.S.J.P. Revalski: Constrained convex optimization problems-well-posedness and stability. Num. Funct. Anal. Optim. **15**, 889–907 (1994)
18. Levitin, E.S., Polyak, B.T.: Convergence of minimizing sequences in conditional extremum problems. Soviet Math. Dokl. **7**, 764–767 (1966)
19. Lignola, M.B., Morgan, J.: Approxiamte solutions and α-well-posedness for variational inequalities and Nash equilibria. In: Zaccour, G. (ed.) Decision and control in management science, pp. 369–378. Kluwer Academic Publishers, Dordrecht (2001)
20. Lignola, M.B., Morgan, J.: α-well-posedness for Nash equilibria and for optimization with Nash equilibrium constraints. J. Glob. Optim. Published Online, 27 June (2006)
21. Loridan, P.: Well-posed vector optimization, recent developments in well-posed variational problems, mathematics and its applications, vol. 331. Kluwer Academic Publishers, Dordrecht (1995)
22. Lucchetti, R.: Well-posedness towards vector optimization. Lecture notes in economics and mathematical systems, vol. 294. Springer-Verlag, Berlin (1987)
23. Lucchetti, R., Revalski, J. (eds.).: Recent developments in well-posed variational problems. Kluwer Academic Publishers, Dordrecht (1995)
24. Margiocco, M., Patrone, F., Chicco, P.: A new approach to Tikhonov well-posedness for Nash equilibria. Optimization **40**, 385–400 (1997)
25. Nguyen, S., Dupuis, C.: An efficient method for computing traffic equilibria in networks with asymmetric transportation costs. Transport. Sci. **18**, 185–202 (1984)
26. Patrone, F.: Well-posedness for Nash equilibria and related topics, recent developments in well-posed variational problems, Math. Appl., vol. 331, pp. 211–227. Kluwer Academic Publishers, Dordrecht (1995)
27. Revalski, J.P.: Hadamard and strong well-posedness for convex programs. SIAM J. Optim. **7**, 519–526 (1997)
28. Tykhonov, A. N.: On the stability of the functional optimization problem. USSR Comput. Math. Math. Phys. **6**, 28–33 (1966)

29. Yamashita, N., Fukushima, M.: On the level-boundedness of the natural residual function for variational inequality problems. Pac. J. Optim. **1**, 625–630 (2005)
30. Yang, H., Yu, J.: Unified approaches to well-posedness with some applications. J. Glob. Optim. **31**, 371–381 (2005)
31. Zhu, D.L.: Augmented Lagrangian theory, duality and decomposition methods for variational inequality problems. J. Optim. Theory Appl. **117**, 195–216 (2003)
32. Zolezzi, T.: Well-posedness criteria in optimization with application to the calculus of variations. Nonlinear Anal. Theory Methods Appl. **25**, 437–453 (1995)
33. Zolezzi, T.: Extended well-posedness of optimization problems. J. Optim. Theory Appl. **91**, 257–266 (1996)
34. Zolezzi, T.: Well-posedness and optimization under perturbations. Ann. Oper. Res. **101**, 351–361 (2001)